Numerical Solutions to Partial Differential Equations

Elliptic Equations

Poisson equation $$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y)$$

Some problems involving this type of equations,

steady-state distribution of heat in a plane region

two-dimensional steady-state problems involving incompressible fluids.

Laplace's equation $$f(x, y) \equiv 0$$

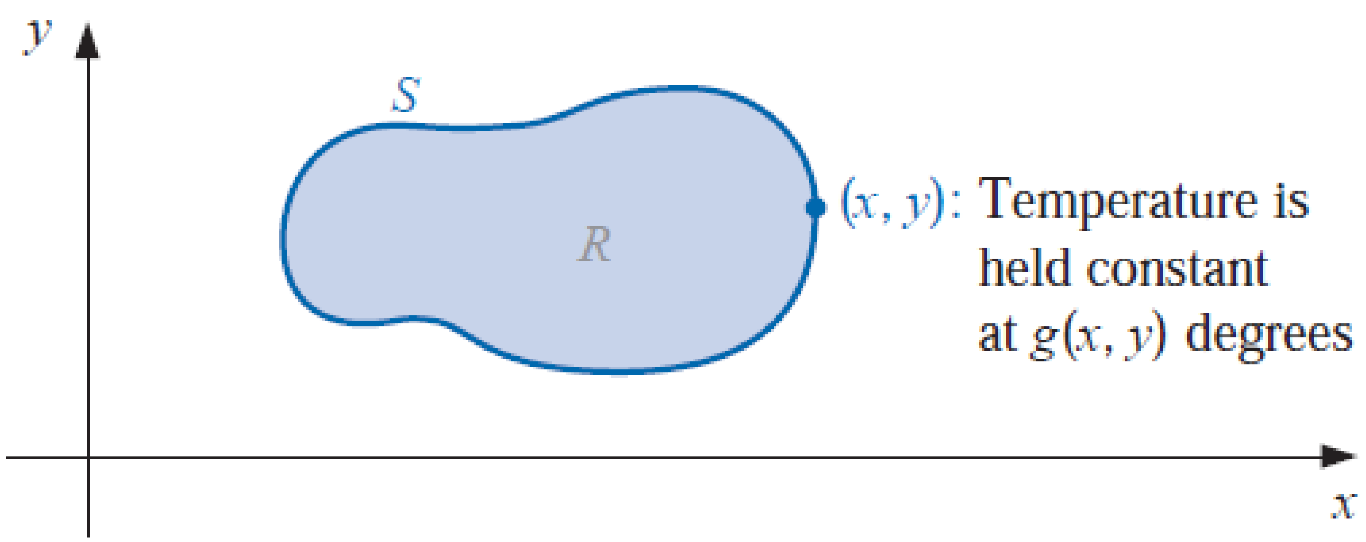
$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0$$

Laplace equation is used to find steady state distribution of temperature in a plane region.

Dirichlet boundary conditions,

$$u(x, y) = g(x, y)$$

for all (x, y) on S, the boundary of the region R.



Parabolic Equations

diffusion equation $$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

The physical problem considered here concerns the flow of heat along a rod of length l which has a uniform temperature within each cross-sectional element. This requires the rod to be perfectly insulated on its lateral surface.

Initial condition,

$$u(x,0) = f(x)$$

Boundary Conditions,

if the ends are held at constant temperatures U_1 and U_2,

$$u(0,t) = U_1 \quad \text{and} \quad u(l,t) = U_2$$

If, instead, the rod is insulated so that no heat flows through the ends,

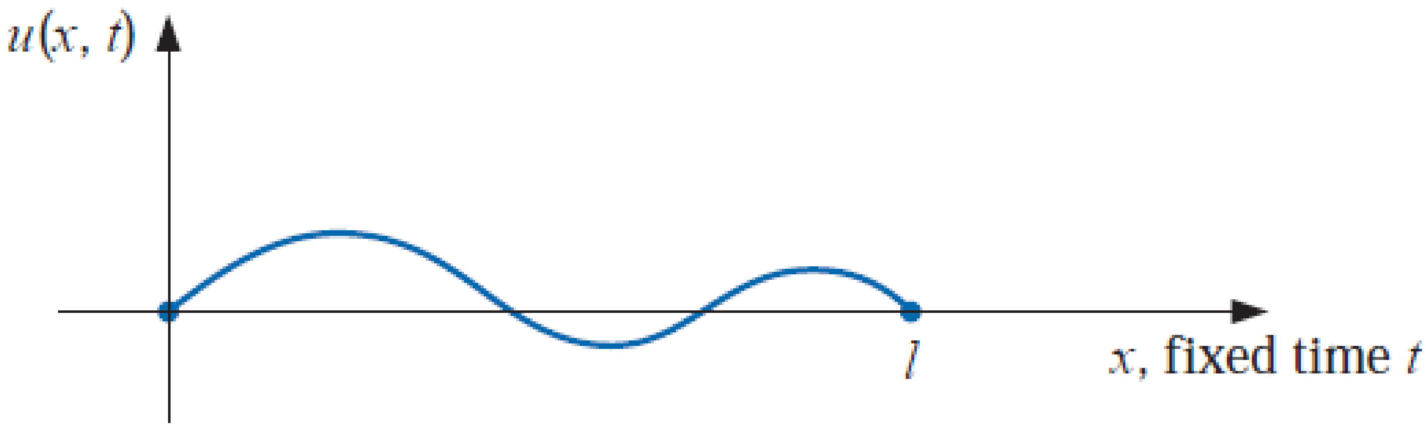
$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(l,t) = 0$$

Hyperbolic Equations

Suppose an elastic string of length l is stretched between two supports at the same horizontal level. If the string is set to vibrate in a vertical plane, the vertical displacement $u(x,t)$ of a point x at time t satisfies the partial differential equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial t^2}(x,t) = 0, \quad \text{for } 0 < x < l \quad \text{and} \quad 0 < t$$

The above equation is called the **wave equation**.



Initial conditions,

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for } 0 \le x \le l$$

Boundary conditions,

If the endpoints are fixed,

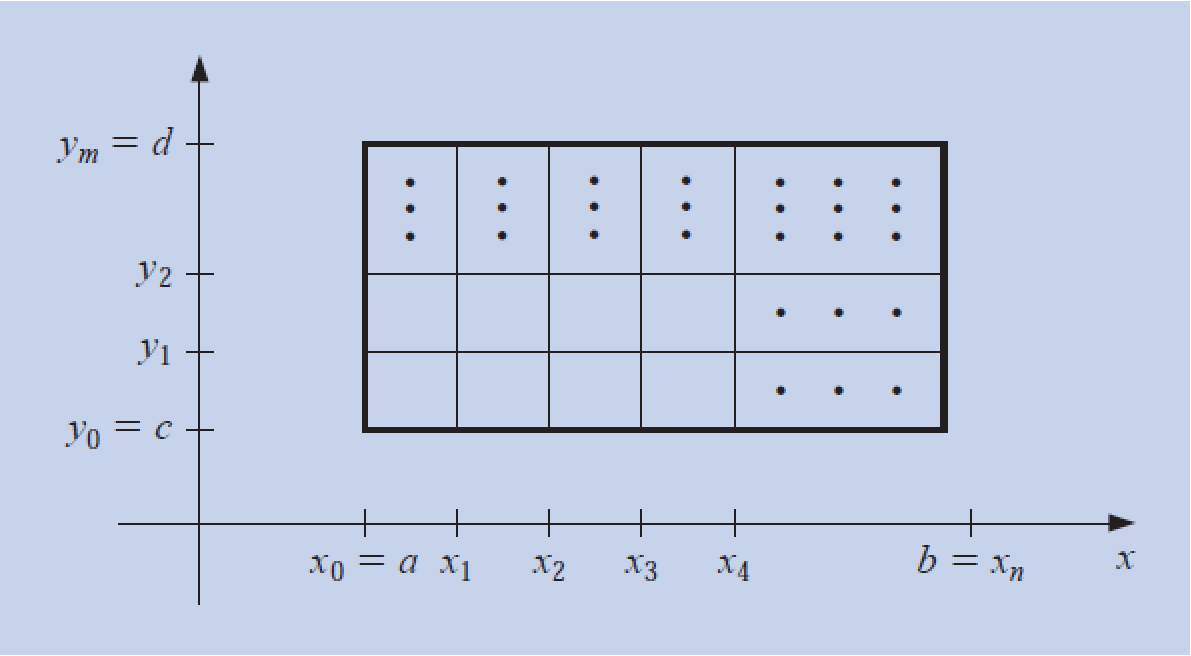
$$u(0, t) = 0 \qquad\qquad u(l, t) = 0$$

Vibrating beam with one or both ends clamped is the other physical problem involving hyperbolic partial differential equation.

Finite-Difference Method

The first step in the numerical solving of a partial differential equation on a domain is the selection of a **grid**. Then, the governing equation must be discretized on the **mesh points**.

Figure shows the mesh points on a rectangular domain. The domain has



been divided into n and m parts in x and y directions, respectively. Step sizes in x and y directions are,

$$h = (b-a)/n \qquad k = (d-c)/m$$

Now, we must discretize equations on the grid points. In fact, it is necessary to approximate derivatives on each mesh point.

Using the Taylor expansion of $u(x)$ about x_i, we have,

$$\begin{aligned} u(x) = u(x_i) + \frac{\partial u}{\partial x}(x_i)(x - x_i) + \frac{\partial^2 u}{\partial x^2}(x_i)\frac{(x - x_i)^2}{2} + \\ \frac{\partial^3 u}{\partial x^3}(x_i)\frac{(x - x_i)^3}{6} + \frac{\partial^4 u}{\partial x^4}(x_i)\frac{(x - x_i)^4}{24} + \ldots \end{aligned} \qquad (1)$$

So,

$$\begin{aligned} u(x_{i+1}) = u(x_i) + \frac{\partial u}{\partial x}(x_i)h + \frac{\partial^2 u}{\partial x^2}(x_i)\frac{h^2}{2} + \\ \frac{\partial^3 u}{\partial x^3}(x_i)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(x_i)\frac{h^4}{24} + \ldots \end{aligned} \qquad (2)$$

Computing the first derivative, we have,

$$\frac{\partial u}{\partial x}\left(x_i\right)=\frac{u(x_{i+1})-u(x_i)}{h}-\frac{\partial^2 u}{\partial x^2}\left(x_i\right)\frac{h}{2}-$$

$$\frac{\partial^3 u}{\partial x^3}\left(x_i\right)\frac{h^2}{6}-\frac{\partial^4 u}{\partial x^4}\left(x_i\right)\frac{h^3}{24}-\ldots$$

We can rewrite the above equation as,

$$\frac{\partial u}{\partial x}\left(x_i\right)=\frac{u(x_{i+1})-u(x_i)}{h}+O(h)$$

O(h) represents the remaining terms including higher order derivatives.

Neglecting this terms, we have,

$$\frac{\partial u}{\partial x}\left(x_i\right)=\frac{u(x_{i+1})-u(x_i)}{h}$$

This is called **forward difference** approximation for the first derivative.

The omitted terms (*O(h)*) form the error of this approximation and is called **truncation error**. The error is of the order of the grid size, *h*.

In a similar fashion, we can use equation (1) to compute $u(x_{i-1})$,

$$u(x_{i-1}) = u(x_i) - \frac{\partial u}{\partial x}(x_i)h + \frac{\partial^2 u}{\partial x^2}(x_i)\frac{h^2}{2} - \frac{\partial^3 u}{\partial x^3}(x_i)\frac{h^3}{6} + \frac{\partial^4 u}{\partial x^4}(x_i)\frac{h^4}{24} + \ldots \qquad (3)$$

We can rewrite the above equation as,

$$\frac{\partial u}{\partial x}(x_i) = \frac{u(x_i) - u(x_{i-1})}{h} + O(h)$$

So, the first derivative can be approximated as,

$$\frac{\partial u}{\partial x}\left(x_i\right) = \frac{u(x_i) - u(x_{i-1})}{h}$$

This is the **backward difference** approximation for the first derivative.

The truncation error is of the first order.

Central difference approximation for the first derivative can be obtained by subtracting equations (2) and (3),

$$\frac{\partial u}{\partial x}\left(x_i\right) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + O(h^2)$$

By omitting the terms including higher order derivatives, $O\left(h^2\right)$,

$$\frac{\partial u}{\partial x}\left(x_i\right) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$

In central approximation, the truncation error is of the second order, $O\left(h^2\right)$. So, this approximation is more accurate rather than forward and Backward approximations.

In a similar fashion, second derivative can be approximated by adding equations (2) and (3),

$$\frac{\partial^2 u}{\partial x^2}\left(x_i\right)=\frac{u(x_{i+1})-2\left(x_i\right)+u(x_{i-1})}{h^2}+O(h^2)$$

By omitting the terms including higher order derivatives, $O\left(h^2\right)$,

$$\frac{\partial^2 u}{\partial x^2}\left(x_i\right)=\frac{u(x_{i+1})-2\left(x_i\right)+u(x_{i-1})}{h^2}$$

The above equation represents the **central difference** approximation for the second derivative.

Solving Elliptic Partial Differential Equations

Example

Determine the steady-state heat distribution in a thin square metal plate with dimensions 0.5 m by 0.5 m using $n = m = 4$. Two adjacent boundaries are held at 0°C, and the heat on the other boundaries increases linearly from 0°C at one corner to 100°C where the sides meet.

Solution

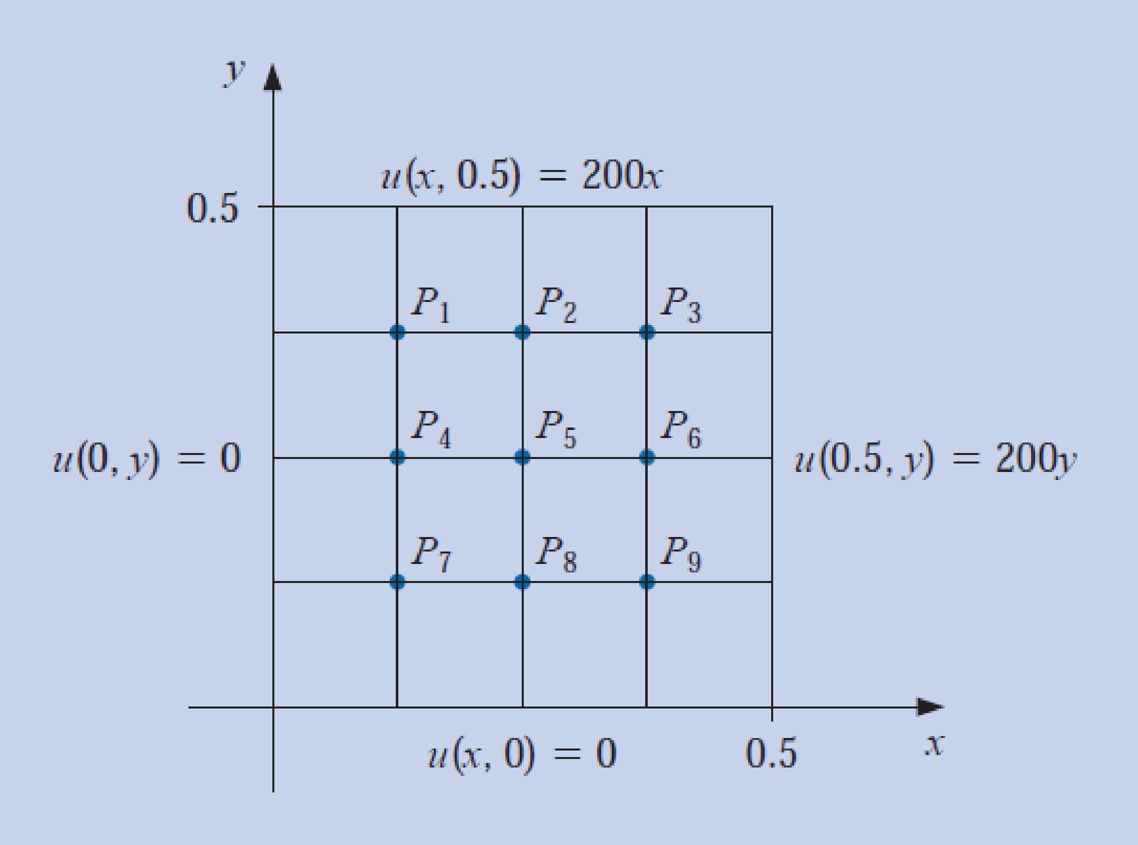
$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0$$

for (x, y) in the set $R = \{ (x, y) \mid 0 < x < 0.5,\ 0 < y < 0.5 \}$.

The boundary conditions are

$$u(0, y) = 0, \quad u(x, 0) = 0, \quad u(x, 0.5) = 200x, \quad \text{and } u(0.5, y) = 200y.$$

If $n = m = 4$, the problem has the grid given in the figure,



Laplace equation can be discretized at node (i,j) as,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

$\Delta x = \Delta y$. So,

$$-u_{i+1,j} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1} - u_{i,j-1} = 0$$

The above equation is valid for each mesh point. Applying this equation for node P_1 results in,

$$-u_2 - 0 + 4u_1 - 200(0.125) - u_4 = 0$$

We can rewrite the above equation as,

$$4u_1 - u_2 - u_4 = 25$$

In a similar fashion, Laplace equation is applied at each mesh point. So, we have a system of equations as,

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9
\end{bmatrix}
=
\begin{bmatrix}
25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25
\end{bmatrix}
$$

The coefficient matrix is a positive definite matrix.

The values of $u_1, u_2, \ldots, u_9$, found by applying the Gauss-Seidel method to this matrix, are given in Table 12.1.

Table 12.1

i	u_i
1	18.75
2	37.50
3	56.25
4	12.50
5	25.00
6	37.50
7	6.25
8	12.50
9	18.75

Choice of Iterative Method

For large systems, an iterative method should be used—specifically, the SOR method. Decomposing the coefficient matrix A as,

$$A = D - L - U$$

The matrix for the Jacobi method can be written as,

$$B = D^{-1}(L + U)$$